UDC 539.3: 534.231.1

SHOCK WAVES IN AN ISOTROPIC ELASTIC SPACE

PMM, Vol. 42, No. 4, 1978, pp. 711-717

A. A. BURENIN and A. D. CHERNYSHOV
(Voronezh)

(Received September 28, 1977)

Shock waves being propagated in an isotropic deformed elastic medium are studied within the framework of adiabatic, quadratic elasticity theory. A system of equations in discontinuities is written down which describes the shockwave propagation process from whose solution the velocities of the possible shocks are determined. Conditions for the existence of possible shocks are obtained from the conditions for solvability of the given system, as a function of the properties of the medium and the deformed state in front of the surface of discontinuities. Some of the results are extended to the case of an arbitrary dependence of the elastic potential on the strain tensor invariants. Constraints on the existence of shocks imposed by the second law of thermodynamics are studied.

An extensive literature (see [1-6], for instance) is devoted to the study of the properties of shocks being propagated in a nonlinear elastic medium. Shocks in an incompressible elastic medium [1] are studied most. Results have been obtained successfuly in the consideration of shocks in a compressible medium for either constraints imposed on the dependence of the elastic potential on the strain tensor invariants [2-4], or on the deformed state ahead of the shock [4-6], or by constraining the analysis just to certain kinds of waves [3, 5, 6]. In this paper no other constraints are introduced, except that taken into account are the highest nonlinear terms (quadratic elasticity theory) and a full study is performed of the properties of shocks in the arbitrary deformed state ahead of the surface of discontinuities.

1. We define an isotropic elastic medium by the elastic potential $W = W(I_1, I_2, I_3)$, where I_1, I_2, I_3 are the Almansi strain tensor components. The Almansi strain tensor components e_{ij} are representable in terms of the displacement vector components u_i in a rectangular Cartesian coordinate system in the form

$$e_{ij} = {}^1/_2\,(u_{i,j} + u_{j,i} - u_{k,i}u_{k,j}), \quad I_1 = e_{jj}, \quad I_2 = e_{ij}e_{ji}, \qquad (1.1)$$
$$I_3 = e_{ij}e_{jk}e_{ki}$$

The stress tensor components σ_{ij} are determined by the formulas

$$\sigma_{ij} = \frac{\rho}{\rho_0}\frac{\partial W}{\partial e_{ik}}(\delta_{kj} - 2e_{kj}) \qquad (1.2)$$
$$\rho/\rho_0 = (1 - 2I_1 + 2I_1^2 - 2I_2 - {}^4/_3 I_1^3 + 4I_1I_2 - {}^8/_3 I_3)^{1/2}$$

where δ_{kj} is the Kronecker symbol, and the ratio between the running density of the medium ρ and the density of the medium in free space ρ_0 is expressed in terms of the strain tensor components by means of the continuity equation.

Let us couple a moving coordinate system x_i $(i = 1, 2, 3)$, to a shock being propagated in the elastic medium by directing the x_1 axis along the normal to the surface of discontinuities. The velocity vector components v_i in the moving coordinate system are evaluated according to the formula

$$v_i = \delta u_i/\delta t + (v_1 - G)u_{i,1} + v_\alpha u_{i,\alpha} \quad (\alpha = 2, 3) \tag{1.3}$$

Here G is the velocity of shockwave propagation $\delta/\delta t$ is the delta derivative with respect to time [7]. Performing the operation of discontinuity in (1.3) and taking into account that the delta-derivative with respect to time of a continuous function is continuous, we obtain

$$[v_i] = u_{i,j}^+ [v_j] + (v_1^- - G)\tau_i, \quad [u_{i,j}] = \tau_i\delta_{1j}, \quad [f] = f^+ - f^- \tag{1.4}$$

The plus and minus signs on the quantities denote that they are evaluated ahead of and directly behind the shock, respectively. We shall henceforth omit the plus signs at the components of the displacement gradient tensor $u_{i,j}$ since unless stipulated otherwise, only componenets of this tensor calculated ahead of the surface of discontinuity will be present everywhere. Solving (1.4) for $[v_j]$, we find

$$[v_1] = \frac{v_1^- - G}{\theta}\{(p_2p_3 - u_{2,3}u_{3,2})\tau_1 + (u_{1,3}u_{3,2} + p_3u_{1,2})\tau_2 + (u_{1,2}u_{2,3} + p_2u_{1,3})\tau_3\} \tag{1.5}$$

$$[v_2] = \frac{v_1^- - G}{\theta}\{(p_1p_3 - u_{1,3}u_{3,1})\tau_2 + (u_{2,3}u_{3,1} + p_3u_{2,1})\tau_1 + (u_{2,1}u_{1,3} + p_1u_{2,3})\tau_3\} \quad (2, 3)$$

$$p_1 = 1 - u_{1,1}, \quad p_2 = 1 - u_{2,2} \quad (2, 3), \quad \theta = p_1p_2p_3 - u_{1,2}u_{2,3}u_{3,1} - u_{2,1}u_{3,2}u_{1,3} - p_1u_{2,3}u_{3,2} - p_2u_{1,3}u_{3,1} - p_3u_{1,2}u_{2,1}$$

Here and henceforth, the system (2.3) will mean that the appropriate unwritten relationship is obtained by commutating the subscripts *2* and *3*.

In this case the dynamic compatibility condition for the discontinuities (the condition of momentum conservation during passage through the shock) can be written in the form

$$V[v_i]/(v_1^- - G) = [\sigma_{i1}], \quad V = \rho^-(v_1^- - G)^2 \tag{1.6}$$

If the quantity $[v_i]$ evalued from (1.5), the $[\sigma_{i1}]$ evaluated according to (1.1) and (1.2) where $W = W(I_1, I_2, I_3)$ are considered known functions in (1.6), then for a given state of strain ahead of the surface of discontinuity the relationship (1.6) is a system of three euqtions in the four unknowns V, τ_i. The parameter V introduced in (1.6) characterizes the propagation velocity of the shock. The system (1.6) can be investigated if one of the discontinuities τ_i is assumed known.

Let us limit ourselves exclusively to second order terms in the components of the displacement gradient tensor in (1.6). To do this, it is sufficient to keep only the following terms

$$W = {}^1/_2\lambda I_1^2 + \mu I_2 + lI_1I_2 + mI_1^3 + nI_3 \tag{1.7}$$

in the Maclauren series expansion of the function $W = W(I_1, I_2, I_3)$

The elastic potential (1.7) is sometimes called the Murnaghan potential, λ, μ are the Lamé parameters, l, m, n are ordinarily called the elastic third order moduli or the Murnaghan coefficients. Substituting (1.7) into (1.2) and limiting ourselves to second order infinitesimals in $u_{i,j}$ we find

$$\begin{gathered}\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + (3m - \lambda)u_{k,k}^2\delta_{ij} + lv_{st}v_{ts}\delta_{ij} + \\ 2(l - \lambda - \mu)\, u_{k,k}v_{ij} + (3n - 4\mu)v_{ik}v_{kj},\ v_{ij} = {}^1/_2\,(u_{i,j} + u_{j,i})\end{gathered} \tag{1.8}$$

If (1.8) is written in discontinuities and the result substituted together with (1.5) into (1.6), we obtain the following system of three equations in the four unknowns V, τ_i:

$$\begin{gathered}(V - T_1)\tau_1 + (b_1V - k_1)\tau_2 + (c_1V - s_1)\tau_3 + \alpha\tau_1^2 + \\ \varkappa\,(\tau_2^2 + \tau_3^2) = 0\end{gathered} \tag{1.9}$$

$$(V - T_2)\tau_2 + (b_2V - k_2)\tau_1 + (c_2V - s_2)\tau_3 + \gamma\tau_1\tau_2 = 0 \qquad (2,3)$$

Here

$$\begin{gathered}T_1 = p_1(\lambda + 2\mu) + \beta\,(u_{2,2} + u_{3,3}) + 2\alpha u_{1,1}, \quad T_2 = p_2\mu + \\ \gamma\,(u_{1,1} + u_{2,2}) + (l - \lambda - \mu)\, u_{3,3} \quad (2,3), \quad \alpha = 3l + 3m + \\ 3n - {}^7/_2\lambda - 7\mu, \quad \beta = 2l + 6m - 4\lambda - 2\mu, \quad \gamma = l + {}^3/_2 n - \\ \lambda - 2\mu, \quad k_1 = (\gamma - 2\mu)\, u_{2,1} + (\gamma + \lambda)\, u_{1,2}\end{gathered}$$

$$\begin{gathered}k_2 = (\gamma - \mu)\, u_{1,2} + \gamma u_{2,1} \quad (2,3), \quad s_1 = (\gamma - 2\mu)\, u_{3,1} + (\gamma + \\ \lambda)\, u_{1,3}\end{gathered}$$

$$\begin{gathered}s_2 = ({}^3/_4 n - \mu)u_{2,3} + ({}^3/_4 n - 2\mu)\, u_{3,2} \quad (2,3), \quad b_1 = u_{1,2}, \ b_2 = \\ u_{2,1} \quad (2,3), \quad c_1 = u_{1,3}, \quad c_2 = u_{2,3} \quad (2,3), \quad \varkappa = {}^1/_2\,(\gamma - 2\mu)\end{gathered}$$

Let us study the possibility of the existence of longitudinal and transverse shocks in the elastic medium.

Setting $\tau_2 = \tau_3 = 0$ in the initial system of equations (1.9), we obtain

$$V = T_1 - \alpha\tau_1, \quad b_2(T_1 - \alpha\tau_1) = k_2 \qquad (2,3) \tag{1.10}$$

A longitudinal shock whose velocity is calculated by the first equality in (1.10) is possible in an elastic medium if the state of strain ahead of the shock satisfies the last conditions of (1.10). Let us note that these conditions are satisfied identically if the state of strain ahead of the surface of discontinuities is such that

$$u_{1,2} = u_{2,1} = 0 \quad (2,3) \tag{1.11}$$

The equalities (1.11) can be considered as sufficient conditions for the existence of longitudinal shocks in the medium.

The transverse shocks $\tau_1 = 0$ are propagated in an elastic medium if the relationships

$$(b_1 V - k_1)\tau_2 + (c_1 V - s_1)\tau_3 + \varkappa(\tau_2^2 + \tau_3^2) = 0 \qquad (1.12)$$
$$(V - T_2)\tau_2 + (c_2 V - s_2)\tau_3 = 0 \qquad (2,3)$$

are satisfied.

The velocity of the transverse shocks should be evaluated by equating the determinant of the homogeneous system of the last equations in (1.12) to zero. The first equality in (1.12) expresses the condition for the existence of transverse shock waves. It must be noted that transverse shocks are impossible in a medium strained in such a way that (1.11) are satisfied. Let (1.11) not be satisfied, but $u_{2,3} = u_{3,2} = 0$, then we obtain from the last equations of (1.12)

$$V_2 = T_2 \quad (2,3) \qquad (1.13)$$

The case under consideration is qualitatively no different from the linear case. Transverse shocks are possible on which $\tau_2 \neq 0$ but $\tau_1 = \tau_3 = 0$ or, conversely, $\tau_3 \neq 0$ but $\tau_2 = \tau_1 = 0$. The nonlinearity is only manifest quantitatively in the propagation velocity values for these shock waves. If $u_{2,3} \neq 0$ or $u_{3,2} \neq 0$, then a transvers shock waves on which τ_2 and τ_3 are not zero simultaneously becomes possible. The velocity of this latter wave is determined by the relationship

$$V_{1,2} = \frac{Q + \{Q^2 - 4(1 - c_2c_3)(T_2T_3 - s_2s_3)\}^{1/2}}{2(1 - c_2c_3)}, \quad Q = T_2 + T_3 + c_2s_3 + c_3s_2 \qquad (1.14)$$

As the influence of the nonlinearity diminishes, the velocity of the longitudinal shock tends to the value $G = \{(\lambda + 2\mu)/\rho_0\}^{1/2}$ for the velocity of a longitudinal shock in linear theory. The velocities of all possible transverse shocks calculated in conformity with (1.13) and (1.14) tend to the value $G = \{\mu / \rho_0\}^{1/2}$ for the velocity of a transverse shock in the linear case as the influence of the nonlinearities diminishes.

The equalities (1.11) are sufficient conditions for the existence of longitudinal shocks. On the other hand, if (1.11) are satisfied, then transverse shocks are impossible in an elastic medium. Let us note that this result carries over even to the case of an arbitrary dependence of the elastic potential on the strain tensor invariants.

Evaluating the stress tensor components from (1.2), let us note that the dependence of σ_{11} on $u_{i,j}$ $(i \neq j)$ is even, i.e., σ_{11} is independent of the sign of $u_{i,j}$ $(i \neq j)$. Therefore, the coefficients of τ_2^{2n-1} and τ_3^{2n-1} in the expression for $[\sigma_{11}]$ in (1.6) will contain the factors $u_{j,1}$ or $u_{1,j}$ $(j \neq 1)$. However, the presence of terms with τ_2^{2n} and τ_3^{2n}, whose coefficients may be independent of the state of strain ahead of the surface of discontinuities, is possible. Therefore, the first equation from (1.6) does not become an identity on a transverse wave. Thus, transverse waves do not exist in an elastic medium if only (1.11) are satisfied ahead of a surface of discontinuity. This latter essentially extends the known result [5] that transverse shocks are impossible in an undeformed elastic medium. Analogously to the above, it can be shown that (1.11) are sufficient conditions for the existence of longitudinal shocks in an isotropic elastic space in the case of an arbitrary dependence of the elastic potential on the Almansi strain tensory invariants.

2. Let us henceforth consider τ_1 to be known and different from zero. Then, by evaluating τ_2 and τ_3 from the last equations in (1.9), and substituting the values obtained into the first equation in (1.9), we obtain the following equation of the fifth power in V:

$$(V - T_1 + \alpha\tau_1) R_1^2(V) + \varkappa\tau_1 \{R_2^2(V) + R_3^2(V)\} - (b_1 V - k_1) R_1(V) R_3(V) - (c_1 V - s_1) R_1(V) R_2(V) = 0 \qquad (2.1)$$

$$R_1(V) = (V - T_2 + \gamma\tau_1)(V - T_3 + \gamma\tau_1) - (c_2 V - s_2)(c_3 V - s_3)$$

$$R_2(V) = (b_3 V - k_3)(V - T_2 + \gamma\tau_1) + (c_3 V - s_3)(b_2 V - k_2) \ (2,3)$$

We seek the solution of (2.1) approximately by considering $u_{1,j}$ and $u_{j,1}$ ($j \neq 1$) to be small quantities. Neglecting squares of these quantities in (2.1), we obtain

$$(V - T_1 + \alpha\tau_1) R_1^2(V) = 0 \qquad (2.2)$$

It follows from (2.2) that

$$V_1^\circ = T_1 - \alpha\tau_1 \qquad (2.3)$$

Substituting (2.3) into (1.9) results in the equalities $\tau_2 = \tau_3 = 0$ in the same approximation. Therefore, the zero approximation for the first root of (2.1) corresponds to the longitudinal wave studied earlier which is propagated in a medium if it is deformed in such a way that (1.11) are satisfied.

Let $u_{i,j} = 0$ $(i \neq j)$ in front of the surface of discontinuity, then we obtain from (2.2)

$$V_2^\circ = T_2 - \gamma\tau_1 \quad (2,3) \qquad (2.4)$$

If the relationships (1.11) are satisfied, but $u_{2,3}$, $u_{3,2}$ or both these components of the displacement gradient tensor are different from zero, then we obtain from (2.2)

$$V_{2,3}^\circ = [2(1 - c_2c_3)]^{-1} \{F \pm [F^2 - 4(1 - c_2c_3)(T_2T_3 - \gamma T_2\tau_1 - \gamma T_3\tau_1 + \gamma^2\tau_1^2 - s_2s_3)]^{1/2}\} \qquad (2.5)$$

$$F = T_2 + T_3 - 2\gamma\tau_1 + c_2s_3 + c_3s_2$$

Therefore, when the state of strain in front of the surface of discontinuity is such that $u_{i,j} = 0$ $(i \neq j)$ shocks are possible in the elastic medium on which either $\tau_1 \neq 0$, $\tau_2 \neq 0$ and $\tau_3 = 0$ or $\tau_1 \neq 0$, $\tau_3 \neq 0$ but $\tau_2 = 0$. When conditions (1.11) are satisfied, and at least one of the components $u_{2,3}$ and $u_{3,2}$ of the displacement gradient tensor is different from zero, then a shock is possible in the elastic medium whose velocity is determined by (2.5). All the components of the wave vector are different from zero on this shock, $\tau_1 \neq 0, \tau_2 \neq 0, \tau_3 \neq 0$. As the influence of the nonlinearities diminishes, the velocities of these latter shocks, defined by (2.4) and (2.5), tend to the value $\{\mu / \rho_0\}^{1/2}$ for the transverse shock velocity in linear theory, hence, these shocks will henceforth be called quasi-transverse.

Upon substituting (2.4) and (2.5) into the system (1.9), we obtain that quasi-transverse shock waves of the kind studied are possible in the medium only if the following relationships are satisfied

$$\tau_2^2 = (V_1^\circ - V_2^\circ)\,\tau_1\,/\,\varkappa, \quad \tau_2^2 + \tau_3^2 = (V_1^\circ - V_2^\circ)\,\tau_1\,/\,\varkappa \qquad (2{,}3) \tag{2.6}$$

It follows from (2.6) that a necessary condition for the existence of quasi-transverse shock waves in an elastic medium whose state of strain satisfies (1.11) will be the requirement that

$$(V_1^\circ - V_2^\circ)\,\tau_1\,/\,\varkappa \geqslant 0 \quad (2{,}3) \tag{2.7}$$

Let l, m, n be negative. This is valid for materials similar to the incompressible (rubber-like) ones since $I_1 \leqslant 0$ for incompressible materials, and hence $l < 0$, $m < 0$, $n < 0$. (Experiments [8] show that l, m, n are negative even for metals). In this case, the requirement (2.7) becomes

$$\tau_1 < 0 \tag{2.8}$$

We note that condition (2.8) also results for positive l, m, n, if their orders are less than the order of λ and μ. The inequality (2.8) is the condition for the existence of quasi-transverse shocks in the case when the third order elastic moduli are zero. In the terminology of plane shocks (2.8) means that quasi-transverse shocks will be simultaneously expansion waves. The order of τ_1 on a quasi-transversewave is of a second order infinitesimal as compared with τ_2 or τ_3.

Let us turn to seeking the next approximation for the roots of (2.1). We note that the first approximation $V_1^{(1)}$ for the first root equals zero since V_1 depends only on even powers of $u_{i,j}$ $(i \neq j)$. Setting $V_1 = V_1^\circ + V_1^{(2)}$, where $V_1^{(2)}$ is a linear combination of $u_{i,j}^2$ $(i \neq j)$ and substituting into (2.1), we find

$$V_1 = V_1^\circ + \varkappa\tau_1\xi^{-2}(\xi_2^2 + \xi_3^2) + \xi^{-1}(a_1\xi_2 + f_1\xi_3) \tag{2.9}$$
$$a_i = b_iV_1^\circ - k_i, \quad f_i = c_1V_1^\circ - s_i, \quad i = 1,\ 2,\ 3$$
$$\xi = (V_1^\circ - V_2^\circ)(V_1^\circ - V_3^\circ) + f_2f_3, \quad \xi_2 = a_3(V_1^\circ - V_2^\circ) + a_2f_3 \quad (2{,}3) \quad |$$

We call a shock wave whose velocity is determined by (2.9) quasi-longitudinal since its velocity tends to the value of the longitudinal shock velocity in linear theory of elasticity as the finfluence of the nonlinearity diminishes.

Setting $V_2 = V_2^\circ + V_2^{(1)}$ $(2{,}3)$ and substituting into (2.1), we obtain

$$V_2 = V_2^\circ + \left\{\frac{\varkappa\tau_1}{V_1^\circ - V_2^\circ}\right\}^{1/2}\left\{\left(\frac{a_3f_3}{V_2^\circ - V_3^\circ}\right)^2 + \left(\frac{\xi_3}{V_2^\circ - V_3^\circ}\right)^2\right\}^{1/2} \quad (2,3) \tag{2.10}$$

We note that the inequality (2.7) and therefore (2.8) as well, have a broader meaning as conditions for the existence of quasi-transverse shocks, than had been remarked. It follows from (2.10) that (2.7) is the condition for the existence of quasi-transverse shocks when conditions (1.11) are not satisfied, i.e., for an arbitrary state of strain ahead of the surface of discontinuity.

3. A shock wave in an adiabatic elastic medium is an irreversible process. The thermodynamic condition of compatibility of the discontinuities, which we write in the form [9]

$$\frac{A}{2}[v_j v_j] + C_j [v_j] - \frac{A}{\rho_0}[W] \geqslant 0 \tag{3.1}$$

$$A = \rho^+ (v_1{}^+ - G), \quad C_j = {}^-\sigma_{i1}{}^+ - A v_i{}^+$$

results from the second law of thermodynamics upon crossing the surface of discontinuity.

Let us recall that a corollary of the second law of thermodynamics for a shock in a perfect gas is the known Zemplen theorem that only compression shocks are possible. In the case under consideration, it is convenient to use (3.1) written in the following form

$$-\frac{V[v_j][v_j]}{2(v_1{}^- - G)} + \sigma_{j1}{}^+ [v_j] - \frac{A}{\rho_0}[W] \geqslant 0 \tag{3.2}$$

Substituting (1.2), (1.5), (1.7) written in discontinuities, and (1.8) evaluated ahead of the shock wave into (3.2) results in the inequality

$$(l + m + n - {}^3/{}_2\lambda - 3\mu)\, \tau_1{}^3 \leqslant 0 \tag{3.3}$$

for a quasi-longitudinal wave when V is defined by (2.9).

Terms not higher than the cubic in the components of the displacement gradient tensor were kept in obtaining (3.3). The inequality (3.3) is the analog of the Zemplen theorem for quasi-longitudinal shocks in an elastic medium. It is hence assumed that higher orders of the components $u_{i,j}$ than the third can be neglected. For negative l, m, n, which ordinarily correspond to specific materials, as has been noted, we obtain from (3.3)

$$\tau_1 \leqslant 0$$

i.e., only quasi-longitudinal compression shocks are possible in such media.

In the case of quasi-transverse shock waves, the inequality (3.2) becomes an identity to the accuracy of cubes in $u_{i,j}$, i.e., energy dissipation on quasi-transverse shocks has a higher order than the third. In this case terms with $u_{i,j}{}^4$ should be taken into account in (1.7). However, if a potential in the form (1.7) is selected in a specific problem, then it should be considered that no quasi-transverse low-intensity shock waves exist in the medium. This latter results from (3.2) after substitution therein of the relations (2.10), (1.5), (1.2) and (1.8), (1.7) written in discontinuities.

REFERENCES

1. Boa-Teh Chu, Transverse shock waves in incompressible elastic solids, J. Mech. and Phys. Solids, Vol. 15, No. 1, 1967.

2. Chernyshov, A. D., On shockwaves propagation in an elastic space with finite deformations, PMM, Vol. 34, No. 5, 1970.

3. Filatov, G. F., On transverse and longitudinal shock waves propagation in an elastic medium, Prikl. Mekhan. Tekhn. Fiz., No. 3, 1972.

4. Burenin, A. A., Nguen Khyu Tkhan', and Chernyshov, A. D., On shock waves propagation in an elastic medium under finite plane

strain, PMM, Vol. 37, No. 5, 1973.

5. Bland, D. R., Non-linear Dynamic Elasticity, Blaisdell Publishing Co., Waltham, Mass.-Toronto-London, Blaisdall Publication Co., 1969.

6. Filatov, G. F., On shock propagation in nonlinear elasticity theory, IN: Collection of Scientific Work, Voronezh University, No. 2, 1971.

7. Thomas, T., Plastic Flow and Rupture in Solids, (Russian translation), "Mir", Moscow, 1964.

8. Zarembo, L. K. and Krasil'nikov, V. A., Introduction to Nonlinear Acoustics, "Nauka", Moscow, 1966.

9 Chernyshov, A. D., On shock waves propagation conditions in media with elastic and plastic properties, IN: Problems of the Mechanics of Mountain Rock, "Nauka", Alma-Ata, 1972.

Translated by M. D. F.